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# A new description of space and time using Clifford multivectors

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**Abstract**

Following the development of the special theory of relativity in 1905, Minkowski proposed a unified space and time structure consisting of three space dimensions and one time dimension, with relativistic effects then being natural consequences of this spacetime geometry. In this paper, we illustrate how Clifford's geometric algebra that utilizes multivectors to represent spacetime, provides an elegant mathematical framework for the study of relativistic phenomena. We show, with several examples, how the application of geometric algebra leads to the correct relativistic description of the physical phenomena being considered. This approach not only provides a compact mathematical representation to tackle such phenomena, but also suggests some novel insights into the nature of time.

*Keywords:* Geometric algebra, Clifford space, Spacetime, Multivectors, Algebraic framework

## 1. Introduction

The physical world, based on early investigations, was deemed to possess three independent freedoms of translation, referred to as the three dimensions of space. This naive conclusion is also supported by more sophisticated analysis such as the existence of only five regular polyhedra and the inverse square force laws. If we lived in a world with four spatial dimensions, for example, we would be able to construct six regular solids, and in five dimensions and above we would find only three [1]. Gravity and the electromagnetic force laws have also been experimentally verified to follow an inverse square law to very high precision [2], indicating the absence of additional macroscopic dimensions beyond three space dimensions. Additionally Ehrenfest [3] showed that planetary orbits are not stable for more than three space dimensions, with Tangherlini [4] extending this result for electronic orbitals around atoms. The importance of the inverse square force law can be seen from Bertrand's theorem from classical mechanics, which states that 'The only

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central forces that result in closed orbits for all bound particles are the inverse-square law and Hooke's law.' [5].

The concept of time is typically modeled as a fourth Euclidean-type dimension appended to the three dimensions of spatial movement forming Minkowski spacetime [6] with spacetime events modeled as four-vectors. Alternative algebraic formalisms though have been proposed such as STA (Space Time Algebra) [7, 8, 9, 10, 11], or APS (Algebra of Physical Space) [12]. The representation of time within the Minkowski framework and also these alternate approaches interpret time as a Cartesian-type dimension. However with the observed non-Cartesian like behavior of time, such as the time axis possessing a negative contribution to the Pythagorean distance, and the observed inability to freely move within the time dimension as is possible with space dimensions suggests that an alternate representation might be preferable. The results of string theory also suggest that the ordinary formulation of physics, in a space-time with three space dimensions and one time dimension, is insufficient to describe our world [13, 14, 15]. Also, Tifft explained the apparent quantization of the cosmological redshift [16, 17] with a model using time with three dimensions [18, 19, 20, 21], additionally the spinorial nature of time noted by another author [22].

Chappell et al. [23] recently employed a two-dimensional multivector as an algebraic model for spacetime in the plane, with time represented as a bivector. A bivector in geometric algebra represents an oriented unit area, which acts as a rotation operator for the plane. Now, through generalizing this two-dimensional multivector description to three-dimensions we produce a three dimensional vector $\mathbf{x}$ and time becomes a unit area oriented within three dimensions, represented as $ic\mathbf{t}$, which can be combined into a multivector $\mathbf{x} + ic\mathbf{t}$. This now gives rise to symmetry between the space and time coordinates, both being represented as three-vectors in our framework, while still producing the results of special relativity. In fact this duality between space and time, with both possessing the same degrees of freedom, is only possible in three dimensions—in two dimensions for example we have two degrees of freedom for translation but only one for rotation [24, 25].

Clifford's geometric algebra was first published in 1873, extending the work of Grassman and Hamilton, creating a single unified real mathematical framework over Cartesian coordinates, which naturally included the algebraic properties of scalars, complex numbers, quaternions and vectors into a single entity, called the multivector [26]. We find that this general algebraic entity, as part of a real three-dimensional Clifford algebra $Cl_{3,0}(\Re)$, provides an equivalent representation to a Minkowski vector space $\Re^{3,1}$ [27, 28], with significant benefits in assisting intuition and in providing an efficient representation.

In order to represent the three independent dimensions of space, Clifford defined three algebraic elements $e_1$, $e_2$ and $e_3$, with the expected unit vector property

$$e_1^2 = e_2^2 = e_3^2 = 1 \tag{1}$$

but with each element anticommuting, that is $e_i e_j = -e_j e_i$, for $i \neq j$. We then find that the composite algebraic element, the trivector $i = e_1 e_2 e_3$ squares to minus one, that is, $i^2 = (e_1 e_2 e_3)^2 = e_1 e_2 e_3 e_1 e_2 e_3 = -e_1 e_1 e_2 e_2 = -1$, assuming an associative algebra, and as it is found to commute with all other elements it can be used interchangeably with the unit imaginary $\mathrm{i} = \sqrt{-1}$. We continue to follow this notational convention throughout the paper, denoting the Clifford trivector with $i$, the scalar imaginary $\mathrm{i} = \sqrt{-1}$ and the

bivector of the plane with an iota, $\iota = e_1e_2$. A general Clifford multivector for three-space can be written by combining all available algebraic elements

$$a + x_1e_1 + x_2e_2 + x_3e_3 + i(t_1e_1 + t_2e_2 + t_3e_3) + ib, \tag{2}$$

where $a$, $b$, $x_k$ and $t_k$ are real scalars, and $i$ is the trivector. We thus have eight degrees of freedom present, in which we use $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$ to represent a Cartesian-type vector, and the bivector $i(t_1e_1 + t_2e_2 + t_3e_3) = t_1e_2e_3 + t_2e_3e_1 + t_3e_1e_2$ to represent time, forming a time vector $\mathbf{t} = t_1e_1 + t_2e_2 + t_3e_3$. Thus in our framework the temporal components are described by the three bivector components of the multivector. However, when we move into the rest frame of the particle, we only require a single bivector component, so that time becomes one dimensional as conventionally assumed. That is $\bigwedge \Re^3$ is the exterior algebra of $\Re^3$ which produces the space of multivectors $\Re \oplus \Re^3 \oplus \bigwedge^2 \Re^3 \oplus \bigwedge^3 \Re^3$, an eight-dimensional real vector space denoted by $Cl_{3,0}(\Re)$, with the complex numbers and quaternions as subalgebras. The presence of the quaternions as a sub-algebra, represented as $a + i\mathbf{t}$, is significant as Hamilton showed that they provide the correct algebra for rotations in three dimensions. The complex numbers using the description in Eq. (2) would be described as $a + ib$ or by using one of the bivectors, such as $a + ie_k$.

It might be argued that four dimensional spacetime is required to describe the Dirac equation, for example, typically described using the four dimensional Dirac matrices as $\gamma^\mu \partial_\mu |\psi\rangle = -\mathrm{i}m|\psi\rangle$, where $\mathrm{i} = \sqrt{-1}$ and $|\psi\rangle$ is an eight dimensional spinor. However it has been shown [29, 30] that it can be reduced to an equivalent equation in three dimensional space using $Cl_{3,0}(\Re)$, being written as

$$\partial\psi = -m\psi^* ie_3, \tag{3}$$

with the multivector gradient operator $\partial = \partial_t + \boldsymbol{\nabla}$, with $\boldsymbol{\nabla}$ being the three-gradient, and $\psi$ the multivector of three dimensions, shown in Eq. (2), and the operation $\psi^*$ flipping the sign of the vector and trivector components. This can also be compared with the well known form of Maxwell's equations in three-space $\partial\psi = J$, with the electromagnetic field represented by the multivector $\psi = \mathbf{E} + i\mathbf{B}$ and sources $J = \rho - \mathbf{J}$. Hence the multivector of $Cl_{3,0}(\Re)$ unifies complex numbers, spinors, four-vectors and tensors into a single algebraic object. We now utilize the basic properties of the Clifford multivector to produce an alternate algebraic representation of space and time.

## Clifford multivector spacetime

It was shown previously [23] that spacetime events can be described with the multivector of two dimensions

$$X = \mathbf{x} + \iota ct, \tag{4}$$

having the appropriate properties to reproduce the results of special relativity, assuming interactions are planar, with $\mathbf{x} = x_1e_1 + x_2e_2$ representing a vector in the plane and $t$ the observer time, and $\iota = e_1e_2$ the bivector of the plane. Squaring the coordinate multivector we find $X^2 = \mathbf{x}^2 - c^2t^2$, thus producing the correct spacetime distance. This representation of coordinates, along with an electromagnetic field $F = \mathbf{E} + \iota cB$, are subject to a Lorentz transformation of the form $L = \mathrm{e}^{\phi\hat{\mathbf{v}}/2}\mathrm{e}^{\iota\theta/2}$, with the first term defining

boosts and the second term rotations, where the transformed coordinates or transformed field are given by $X' = LXL^\dagger$, where $L^\dagger = \mathrm{e}^{-\iota\theta/2}\mathrm{e}^{-\phi\hat{\mathbf{v}}/2}$. That is, a combined boost and a rotation can be written as

$$X' = \mathrm{e}^{\phi\hat{\mathbf{v}}/2}\mathrm{e}^{\iota\theta/2}X\mathrm{e}^{-\iota\theta/2}\mathrm{e}^{-\phi\hat{\mathbf{v}}/2}. \tag{5}$$

We can see that the multivector in Eq. (4) essentially sets up two orthogonal directions $\mathbf{x}$ and $\iota = ie_3$, and so we can generalize this structure to three dimensions through a general rotation of the $e_1e_2$ plane into three-space forming the three-dimensional multivector

$$X = \mathbf{x} + ic\mathbf{t}, \tag{6}$$

where $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$ is the coordinate vector and $\mathbf{t} = t_1e_1 + t_2e_2 + t_3e_3$ a time vector, with $i = e_1e_2e_3$ the trivector. To maintain the orthogonality implied by the multivector in two dimensions we therefore have the constraints $\mathbf{x} \cdot \mathbf{t} = 0$. Now, squaring the multivector we find

$$\begin{aligned} X^2 &= (\mathbf{x} + ic\mathbf{t})(\mathbf{x} + ic\mathbf{t}) \\ &= \mathbf{x}^2 - c^2\mathbf{t}^2 + 2ic\mathbf{x} \cdot \mathbf{t} = \mathbf{x}^2 - c^2\mathbf{t}^2 \end{aligned} \tag{7}$$

using the fact that $\mathbf{xt} + \mathbf{tx} = 2\mathbf{x} \cdot \mathbf{t} = 0$, referring to Eq. (A.3) in the Appendix, thus producing the correct spacetime distance in three dimensions. Thus an event can be described by a position vector in three dimensions, and a three dimensional time vector in our framework, and to maintain Lorentz invariance we require these two vectors to be orthogonal. Hence Einstein's assertion of the invariant distance in Eq. (7) in his theory of special relativity, from this new viewpoint, becomes simply an assertion of the orthogonality of the space and time three-vectors describing an event in Eq. (6).

We have from Eq. (4) the multivector differential

$$dX = d\mathbf{x} + icd\mathbf{t}. \tag{8}$$

For the rest frame of the particle we have $dX_0^2 = -c^2d\tau^2$, which defines the proper time $\tau$ of the particle. In the co-moving frame of the particle the time vector does not require three components with which to describe its orientation, but only has one degree of freedom defining its length. We have assumed that the speed of light $c$ is the same in the rest and the moving frame, as required by Einstein's second postulate. Now, if the spacetime distance defined in Eq. (7) is invariant under the Lorentz transformations defined later in Eq. (16), then we can equate the rest frame interval to the moving frame interval, giving

$$c^2d\tau^2 = c^2d\mathbf{t}^2 - d\mathbf{x}^2 = c^2d\mathbf{t}^2 - \mathbf{v}^2d\mathbf{t}^2 = c^2d\mathbf{t}^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right), \tag{9}$$

assuming $d\mathbf{x}^2 = \mathbf{v}^2d\mathbf{t}^2$, a vector equation that we confirm shortly. Taking the square root of Eq. (9), we find the time dilation formula $|d\mathbf{t}| = \gamma d\tau$ where

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{10}$$

This equation implies that the length of the time vector $|d\mathbf{t}|$, in this formalism, equates to the scalar time variable $t$ typically employed in special relativity. From Eq. (8), we can now calculate the proper velocity through differentiating with respect to the scalar proper time, giving a multivector representing velocity

$$U = \frac{dX}{d\tau} = \frac{d\mathbf{x}}{|d\mathbf{t}|}\frac{|d\mathbf{t}|}{d\tau} + ic\frac{d\mathbf{t}}{d\tau} = \gamma(\mathbf{v} + i\mathbf{c}), \tag{11}$$

where we use $\frac{|d\mathbf{t}|}{d\tau} = \gamma$ and $\mathbf{v} = \frac{d\mathbf{x}}{|d\mathbf{t}|}$ and the speed of light now becomes vectorial in the direction of the time vector $\mathbf{c} = c\hat{\mathbf{t}}$. We can then find

$$U^2 = \gamma^2(\mathbf{v} + i\mathbf{c})^2 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1}(\mathbf{v}^2 - \mathbf{c}^2 + 2i\mathbf{v}\cdot\mathbf{c}) = -c^2, \tag{12}$$

with the orthogonality of space and time $\mathbf{x}\cdot\mathbf{t} = 0$, implying $\mathbf{v}\cdot\mathbf{c} = 0$. We define the momentum multivector

$$P = mU = \gamma m\mathbf{v} + i\gamma m\mathbf{c} = \mathbf{p} + i\frac{\mathbf{E}}{c}, \tag{13}$$

with the relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$ and the total energy now having a vectorial nature $\mathbf{E} = \gamma mc\mathbf{c}$.

Now, as $U^2 = -c^2$, then $P^2 = -m^2c^2$ is an invariant describing the conservation of momentum and energy, which gives

$$P^2c^2 = \mathbf{p}^2c^2 - \mathbf{E}^2 = -m^2c^4, \tag{14}$$

or $\mathbf{E}^2 = m^2c^4 + \mathbf{p}^2c^2$, the relativistic expression for the conservation of momentum-energy, using the orthogonality of momentum and energy from $\mathbf{p}\cdot\mathbf{E} = \gamma^2m^2c(\mathbf{v}\cdot\mathbf{c}) = 0$. The vectorial nature of energy, analogous to the Poynting vector describing energy flow, is possible because for a Clifford vector $\mathbf{E}$, we find that $\mathbf{E}^2$ is in general a scalar quantity and so can satisfy the Einstein energy-momentum relation as shown. At rest the energy $\mathbf{E_0} = mc\mathbf{c}$ has a vectorial nature and hence the Einstein momentum-energy relation in Eq. (14) can be naturally interpreted as a Pythagorean triangle relation between a constant rest energy vector $mc\mathbf{c}$ and the momentum vector $\mathbf{p}c$.

### *1.1. The Lorentz Group*

The Lorentz transformations describe the transformations for observations between inertial systems in relative motion. The set of transformations describing rotations and boosts connected with the identity is referred to as the restricted Lorentz group $SO^+(3,1)$. We find that the exponential of the bivector $\mathrm{e}^{i\hat{\mathbf{u}}\theta}$, describes rotations in the plane $i\hat{\mathbf{u}}$, as shown in Eq. (B.3), however, more generally, we can define the exponential of a full multivector $M$ defined as in Eq. (2), by constructing the Taylor series

$$\mathrm{e}^M = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \ldots, \tag{15}$$

which is absolutely convergent for all multivectors $M$ [31]. We find that the exponential of a pure vector $\mathrm{e}^{\phi\hat{\mathbf{v}}}$ describes boosts, and so if we define the combined operator consisting of a boost and rotation

$$L = \mathrm{e}^{\phi\hat{\mathbf{v}}}\mathrm{e}^{i\hat{\mathbf{w}}\theta}, \tag{16}$$

where $\hat{\mathbf{v}}$ is the boost direction and $\hat{\mathbf{w}}$ is the rotation axis, we find that operations of this form will leave the spacetime interval given by $X^2$ invariant, defined in Eq. (7), and defines the homogeneous Lorentz group. Defining the dagger operation $L^\dagger = \mathrm{e}^{-i\hat{\mathbf{w}}\theta}\mathrm{e}^{-\phi\hat{\mathbf{v}}}$, we find $LL^\dagger = 1$ and hence the transformed coordinates

$$X'^2 = LXL^\dagger LXL^\dagger = LX^2L^\dagger = X^2, \tag{17}$$

using associativity and the fact that $X^2$ is a scalar as shown in Eq. (7), and so is unaffected by boosts and rotations. Also of interest are transformations of the form

$$\mathrm{e}^{\phi\hat{\mathbf{v}}+i\hat{\mathbf{w}}\theta}, \tag{18}$$

which will also leave the spacetime interval invariant and efficiently describe the Thomas rotation. It can be seen that Eq. (16) is a very efficient description of a general Lorentz transformation which we will find applies to both particles and fields, which normally requires the definition two separate transformations described using $4 \times 4$ matrices.

*1.2. Spacetime boosts*

Using boost operators of the form $\mathrm{e}^{\phi\hat{\mathbf{v}}}$, where the vector $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3 \mapsto \phi\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is a unit vector, with $\hat{\mathbf{v}}^2 = 1$, we find

$$\mathrm{e}^{\phi\hat{\mathbf{v}}} = 1 + \phi\hat{\mathbf{v}} + \frac{\phi^2}{2!} + \frac{\phi^3\hat{\mathbf{v}}}{3!} + \frac{\phi^4}{4!} + \cdots = \cosh\phi + \hat{\mathbf{v}}\sinh\phi. \tag{19}$$

The boost is then defined in terms of the rapidity $\phi$ through $\tanh\phi = |\mathbf{v}|/c$ that can be rearranged to give $\cosh\phi = \gamma$ and $\sinh\phi = \gamma|\mathbf{v}|/c$. Hence $\mathrm{e}^{\phi\hat{\mathbf{v}}} = \gamma\,(1 + \mathbf{v}/c)$.

We define the transformation for boosting fields $F$ as

$$F' = \mathrm{e}^{-\hat{\mathbf{v}}\phi/2}F\mathrm{e}^{\hat{\mathbf{v}}\phi/2}. \tag{20}$$

We define an electromagnetic field multivector $F = \mathbf{E} + ic\mathbf{B}$, which combines the $\mathbf{E}$ and $\mathbf{B}$ fields into a single object and has the advantage that it represents their distinct properties as polar and axial vectors. For an example of a Lorentz boost of a field, we select a pure electric field given by $\mathbf{E} = E_xe_1 + E_ye_2 + E_ze_3$. Then applying a boost according to Eq. (20), using as an example a boost $\mathbf{v} = ve_1$ in the $e_1$ direction, we find

$$\begin{aligned}
\mathrm{e}^{-\frac{\hat{\mathbf{v}}\phi}{2}}\mathbf{E}\mathrm{e}^{\frac{\hat{\mathbf{v}}\phi}{2}} &= \left(\cosh\frac{\phi}{2} - e_1\sinh\frac{\phi}{2}\right)(E_xe_1 + E_ye_2 + E_ze_3) \\
&\times \left(\cosh\frac{\phi}{2} + e_1\sinh\frac{\phi}{2}\right) \\
&= E_xe_1 + (E_ye_2 + E_ze_3)(\cosh\phi + e_1\sinh\phi) \\
&= E_xe_1 + \gamma(E_ye_2 + E_ze_3) + i\frac{\gamma v(-E_ye_3 + E_ze_2)}{c},
\end{aligned} \tag{21}$$

which are the correct Lorentz transformations for an electromagnetic field. That is, the parallel field is unaffected, the perpendicular field $E_y$ and $E_z$ has been increased to $\gamma E_y$ and $\gamma E_z$ and the term $i\frac{\gamma v(-E_ye_3+E_ze_2)}{c}$ is a bivector, and hence this term gives the expected induced magnetic field, with $B_y = \frac{\gamma vE_z}{c}$ and $B_z = -\frac{\gamma vE_y}{c}$.

We apply the boost of coordinates through the transformation in Eq. (20), but replacing $F$ with the spacetime coordinates $X$, but acting with a vector perpendicular to the boost velocity $\mathbf{v}$, that is

$$X' = \mathrm{e}^{-\hat{\mathbf{v}}^{\perp}\phi/2} X \mathrm{e}^{\hat{\mathbf{v}}^{\perp}\phi/2}. \tag{22}$$

For example, for the case of a velocity vector $\mathbf{v} = ve_1$ relative to a position vector $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$, we have $\mathbf{v}^{\perp} = \frac{v}{\sqrt{x_2^2+x_3^2}}(x_2e_2 + x_3e_3)$. In order to recover the conventional results for the Lorentz boost we will also need to set $\mathbf{t}$ perpendicular to $\mathbf{v}^{\perp}$, or $\hat{\mathbf{t}} = \frac{1}{\sqrt{x_2^2+x_3^2}}(x_3e_2 - x_2e_3)$. We then have the transformed spacetime coordinates

$$\begin{aligned} X' &= \mathrm{e}^{-\frac{\hat{\mathbf{v}}^{\perp}\phi}{2}}(\mathbf{x} + ic\mathbf{t})\mathrm{e}^{\frac{\hat{\mathbf{v}}^{\perp}\phi}{2}} \\ &= x_1e_1\mathrm{e}^{\phi\hat{\mathbf{v}}^{\perp}} + x_2e_2 + x_3e_3 + ic\mathbf{t}\mathrm{e}^{\phi\hat{\mathbf{v}}^{\perp}} \\ &= x_1e_1\gamma\left(1 + \mathbf{v}^{\perp}/c\right) + x_2e_2 + x_3e_3 + ict\hat{\mathbf{t}}\gamma\left(1 + \mathbf{v}^{\perp}/c\right) \\ &= \gamma(x_1 - vt)e_1 + x_2e_2 + x_3e_3 + i\gamma\left(ct - \frac{vx_1}{c}\right)\hat{\mathbf{t}}, \end{aligned} \tag{23}$$

the correct Lorentz boost of coordinates, though with time still having a vectorial nature and remaining orthogonal to $\mathbf{x}$ as required.

Hence the exponential map of the vector components of the multivector, naturally produces the correct Lorentz boost transformation of the electromagnetic field and spacetime coordinates.

### *1.3. Energy and momentum conservation*

It has been claimed that having multiple time dimensions will violate the law of conservation of energy [32]—in our case with a time represented as a three vector, this argument does not apply as we maintain orthogonality between the momentum and energy vectors. As an example of its use we analyze Compton scattering, and recover the standard Compton formula. It is well established experimentally that energy and momentum conservation applies in relativistic dynamics, provided we define the momentum as $\gamma m\mathbf{v}$, and the energy as $\gamma mc^2$ [33], which becomes in our case $\gamma mc\mathbf{c}$. We now find, however, that these two conservation laws can be bundled into a single conservation law, the conservation of the momentum multivector defined in Eq. (13).

For example, if we are given a set of isolated particles which are involved in an interaction, which then produce another set of particles as output, then, in order to describe this interaction we include a separate momentum multivector $P$ for each particle, equating the initial and final states with

$$\sum P_{\text{initial}} = \sum P_{\text{final}}. \tag{24}$$

We know $\mathbf{E} = |\mathbf{p}|\mathbf{c}$ for a massless particle, so using Eq. (13) we would write the momentum multivector for a photon as $\Gamma = \mathbf{p} + i|\mathbf{p}|\hat{\mathbf{c}} = \mathbf{p} + i\mathbf{p}^{\perp}$, which gives $\Gamma^2 = 0$ and for a massive particle $P^2 = -m^2c^2$ as shown in Eq. (14).

Compton scattering involves an input photon striking an electron at rest with the deflected photon and moving electron as products. We can then write energy and momentum conservation as conservation of multivectors between initial and final states as

$\Gamma_i + P_i = \Gamma_f + P_f$, which can be rearranged to $(\Gamma_i - \Gamma_f) + P_i = P_f$. Squaring both sides we find

$$(\Gamma_i - \Gamma_f)^2 + P_i(\Gamma_i - \Gamma_f) + (\Gamma_i - \Gamma_f)P_i + P_i^2 = P_f^2, \tag{25}$$

remembering that in general the multivectors do not commute. Now, we have the generic results that $P_i^2 = P_f^2 = -m^2c^2$ and $\Gamma_i^2 = \Gamma_f^2 = 0$ so that we can simplify

$$\begin{aligned}(\Gamma_i - \Gamma_f)^2 &= \Gamma_i^2 + \Gamma_f^2 - \Gamma_i\Gamma_f - \Gamma_f\Gamma_i \\ &= -2\left(\mathbf{p}_i \cdot \mathbf{p}_f + i(\mathbf{p}_i^\perp \cdot \mathbf{p}_f + \mathbf{p}_i \cdot \mathbf{p}_f^\perp) - \mathbf{p}_i^\perp \cdot \mathbf{p}_f^\perp\right).\end{aligned} \tag{26}$$

Now, because the vector direction of the energy given by $\mathbf{p}^\perp$ are out of the plane defined by the input and output photon, then $\mathbf{p}_i \cdot \mathbf{p}_f^\perp = 0$ and $\mathbf{p}_i^\perp \cdot \mathbf{p}_f = 0$ and $\mathbf{p}_i^\perp \cdot \mathbf{p}_f^\perp = |\mathbf{p}_i||\mathbf{p}_f|$, and so

$$(\Gamma_i - \Gamma_f)^2 = -2(\mathbf{p}_i \cdot \mathbf{p}_f + |\mathbf{p}_i||\mathbf{p}_f|) = 2|\mathbf{p}_i||\mathbf{p}_f|(1 - \cos\theta). \tag{27}$$

For the second and third terms in Eq. (25), using $P_i = im\mathbf{c}$ as the initial electron has no momentum, we have $mi(\mathbf{c}(\Gamma_i - \Gamma_f) + (\Gamma_i - \Gamma_f)\mathbf{c}) = -2mc(|\mathbf{p}_i| - |\mathbf{p}_f|)$. We therefore find from Eq. (25) that

$$|\mathbf{p}_i||\mathbf{p}_f|(1 - \cos\theta) - mc(|\mathbf{p}_i| - |\mathbf{p}_f|) = 0. \tag{28}$$

Dividing through by $|\mathbf{p}_i||\mathbf{p}_f|$ and substituting $|\mathbf{p}| = \frac{h}{\lambda}$ we recover Compton's well known formula

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta). \tag{29}$$

The advantage of the momentum multivector is that energy and momentum are two components of a single multivector and so it allows a more direct solution path as the momentum and energy do not need to be treated separately.

### *1.4. Wave mechanics*

Employing the de Broglie relations $\mathbf{p} = \hbar\mathbf{k}$ and $\mathbf{E} = \hbar\mathbf{w}$, where the equation for energy now becomes a vector equation analogous to the equation for momentum, we find using Eq. (13), the wave multivector

$$K = \frac{P}{\hbar} = \mathbf{k} + i\frac{\mathbf{w}}{c}, \tag{30}$$

where the direction of the vector $\mathbf{w}$ represents an implied axis of rotation at the de Broglie frequency for the particle, lying perpendicular to the momentum vector $\hbar\mathbf{k}$, so that we have $\mathbf{k} \cdot \mathbf{w} = \frac{1}{\hbar^2}\mathbf{p} \cdot \mathbf{E} = 0$. We then find

$$K^2 = \mathbf{k}^2 - \frac{\mathbf{w}^2}{c^2} = -\frac{1}{\lambda_c^2}, \tag{31}$$

giving the dispersion relation for a wave that is relativistically invariant, where $\lambda_c = \frac{\hbar}{mc}$ is the reduced Compton wave length.

The dot product of the wave and spacetime multivectors $K \cdot X = \mathbf{k} \cdot \mathbf{x} - \mathbf{w} \cdot \mathbf{t}$ gives the phase of a traveling wave. For a plane wave we can therefore write $\psi = \mathrm{e}^{iK \cdot X}$, which leads to the operators $\mathbf{p} = -i\hbar\nabla_x$ and $\mathbf{E} = i\hbar\nabla_t$, where $\nabla_t = \frac{1}{c}\left(e_1\partial_r + e_2\partial_s + e_3\partial_t\right)$

is the gradient operator over three dimensions of time, labeled $r, s, t$, and $\nabla_x = e_1\partial_x + e_2\partial_y + e_3\partial_z$ is the conventional three-gradient over space. We now have momentum and energy operators with complete symmetry between space and time except for the sign difference. We therefore find $P^2 = \hbar^2\left(\nabla_t^2 - \nabla_x^2\right)$ giving a differential operator with three dimensions of time analogous to the d'Alembertian. However using $P^2 = -m^2$ we can write

$$P^2\psi = \left(\nabla_t^2 - \nabla_x^2\right)\psi = -\frac{m^2c^2}{\hbar^2}\psi, \tag{32}$$

an equation analogous to the Klein-Gordon equation. We can also immediately identify a plane wave solution as $\psi = \mathrm{e}^{iK\cdot X}$. For a stationary state solution $\psi(\mathbf{r},\mathbf{t}) = \phi(\mathbf{r})\mathrm{e}^{-i\mathbf{w}\cdot\mathbf{t}}$, with a fixed vector energy $\mathbf{E} = \hbar\mathbf{w}$, we produce the equation $-\nabla_x^2\phi(\mathbf{r}) = (\mathbf{E}^2 - \frac{m^2c^2}{\hbar^2})\phi(\mathbf{r})$, isomorphic to the conventional stationary state solutions for the Klein-Gordon equation.

Following Dirac's original strategy [34] of factorizing the relativistic energy-momentum relation in Eq.(14) we find, remembering that we defined previously $\mathbf{p}\cdot\mathbf{E} = 0$,

$$\mathbf{p}^2 - \frac{\mathbf{E}^2}{c^2} + m^2c^2 = \left(\mathbf{p} + i\frac{\mathbf{E}}{c} + imc\right)\left(\mathbf{p} + i\frac{\mathbf{E}}{c} - imc\right) = (P + imc)(P - imc) = 0, \tag{33}$$

a factorization analogous to Diracs' but not requiring the use of matrices. Alternatively, starting from the Klein-Gordon equation, we can seek an equation

$$P\psi = \psi^* M, \tag{34}$$

where $M$ is a multivector, and the star operation is a possible automorphism. Using the associativity of geometric algebra, we can then write $P^2\psi = P(P\psi) = (P\psi^*)M = \psi M^2$, thus satisfying the Klein-Gordon equation, provided $M^2 = -\frac{m^2c^2}{\hbar^2}$. Thus Eq. (34) is in the form of the Dirac equation, which can be compared with the conventional Dirac equation with a single time dimension in Eq. (3). The conventional Dirac wave function consists of bi-spinors that possesses eight degrees of freedom and so can be made isomorphic to a wave function $\psi$ represented by the three-space multivector shown in Eq. (2), also possessing eight degrees of freedom. For the multivector $\psi$, we can then define $J = \psi\tilde{\psi} = \rho + \mathbf{J}$, giving a positive definite scalar $\rho$, as required for a probability density, and a current $\mathbf{J}$. The further ramifications of these wave equations involving time as a bivector should now be further investigated.

### *1.5. Time in Schrödinger's equation*

Schrödinger's equation can be written in geometric algebra by replacing the scalar unit imaginary $\sqrt{-1}$ with the bivector $\iota = ie_3 = e_1e_2$, giving

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = \iota\hbar\partial_t\psi, \tag{35}$$

with the wave function then given by the multivector $\psi = a + \iota b$. Because $\iota$ squares to minus one and commutes with scalars it acts equivalently to the conventional scalar imaginary. For a massive particle described by a Gaussian wave packet in momentum space $A(p) = e^{-(p-p_0)^2/\sigma^2}$, also known as the minimum uncertainty wave packet, we

find the position space wave function $\psi(x,t)=\int A(p)e^{\iota(px-E(p)t)/\hbar}dp$, where in the non-relativistic case we have $E(p)=\frac{p^2}{2m}$, which gives

$$\psi(x',t')=\frac{\sigma}{\pi^{1/4}\sqrt{A}(1+t'^2)^{1/4}}\mathrm{e}^{-\frac{A^2(x'-t')^2}{2(1+t'^2)}}\mathrm{e}^{\frac{\iota A^2\left(2x'-t'(1-x'^2)\right)}{2(1+t'^2)}}\mathrm{e}^{-\frac{\iota}{2}\arctan t'}, \tag{36}$$

where we have scaled the variables such that $x'=\frac{\sigma^2}{k_0}x$, $t'=\frac{\sigma^2\hbar}{m}t$ and $A=\frac{k_0}{\sigma}$. Centering our observation at the expected position using $x'_m=x'-t'$, we find

$$\psi(x',t')=\frac{\sigma}{\pi^{1/4}\sqrt{A}(1+t'^2)^{1/4}}\mathrm{e}^{-\frac{A^2x_m'^2}{2(1+t'^2)}}\mathrm{e}^{\frac{\iota A^2}{2}\left(t'+2x'_m+t'x_m'^2/(1+t'^2)\right)}\mathrm{e}^{-\frac{\iota}{2}\arctan t'}. \tag{37}$$

Switching to unscaled variables we find

$$\psi(x,t)=\frac{\sigma^{3/2}\sqrt{m/k_0}}{\pi^{1/4}(m^2+\sigma^4\hbar^2t^2)^{1/4}}\mathrm{e}^{-\frac{\sigma^2m^2x_m^2}{2(m^2+\sigma^4\hbar^2t^2)}}\mathrm{e}^{\frac{\iota\hbar k_0^2t}{2m}}\mathrm{e}^{\iota k_0x_m}\mathrm{e}^{\frac{\iota m\hbar t\sigma^4x_m^2}{2(m^2+\sigma^4\hbar^2t^2)}}\mathrm{e}^{-\frac{\iota}{2}\arctan\frac{\hbar\sigma^2t}{m}}. \tag{38}$$

We now define $w_0=\frac{E_0}{\hbar}=\frac{\hbar^2k_0^2}{2m}$ and a normalization factor $N=\frac{\sigma^{3/2}\sqrt{m/k_0}}{\pi^{1/4}(m^2+\sigma^4\hbar^2t^2)^{1/4}}$. Then, retaining the non-oscillatory terms in space, about the point $x_m=0$, we have

$$\psi(x,t)=N\mathrm{e}^{-\frac{x_m^2}{s^2(t)}}\mathrm{e}^{\iota w_0t}, \tag{39}$$

where we have written $s(t)=\frac{\sqrt{2(m^2+\sigma^4\hbar^2t^2)}}{\sigma m}$, which represents the spread of the Gaussian distribution (representing the probability amplitude of the particle) in space. We also neglect the term $\mathrm{e}^{-\frac{\iota}{2}\arctan\frac{\hbar\sigma^2t}{m}}$, which rapidly converges to $\mathrm{e}^{-\frac{\iota\pi}{4}}$ for a narrow wave packet in space and large time $t$.

Inspecting Eq. (39) we note firstly a Gaussian distribution which is expanding approximately linearly with time, for large $t$ and a term $\mathrm{e}^{\iota w_0t}$ representing a steady rotation in time at the de Broglie frequency $w_0$, as shown by Eq. (5). This result, with time associated with a bivector in Eq. (39), thus supports our description of time, shown in Eq. (4).

## Discussion

We show that the Clifford multivector, given in Eq. (4), provides an alternative representation for spacetime, but allowing us to produce the results of special relativity directly from $Cl_{3,0}(\Re)$. In this formulation, as shown in Eq. (6), time is encoded by a bivector (also called an axial or a pseudovector) and space by a polar vector. Although time locally remains one dimensional in the particles' rest frame, when the Clifford multivector is transformed into a moving observer frame, as a consequence of $Cl_{3,0}(\Re)$ space, proper time decomposes into three components. In this moving frame, these three components are formally incorporated into a multivector as a complex-like number $\mathbf{x}+ic\,\mathbf{t}$, where $\mathbf{x}$ and $\mathbf{t}$ are orthogonal three-vectors and $i$ is the trivector. Our approach thus avoids the use of a mixed metric, creating the negative contribution of time to the invariant distance $\mathbf{x}^2-c^2\,\mathbf{t}^2$, through the use of the trivector $i=e_1e_2e_3$. Previous approaches have used

the unit imaginary $\mathrm{i} = \sqrt{-1}$ for this purpose, but our approach has the advantage of retaining a real-valued space. This description also provides new insight into the nature of time, now being represented as a bivector in three dimensional space. Other physical quantities described by bivectors (or axial vectors) include the magnetic field, torque and angular momentum. These quantities each have a rotational nature, and inspecting the local spacetime event $\mathbf{x} + ic\,\mathbf{t}$, we can interpret $\mathbf{x}$ as the three translational freedoms and $ic\,\mathbf{t}$ as the three rotational freedoms of physical space. This interpretation of time was also confirmed by our investigation of the Schrödinger's equation describing a single massive particle in Section 1.5.

This framework also has the benefit that the Lorentz transformation becomes $\mathrm{e}^{\phi\hat{\mathbf{v}}}\mathrm{e}^{\iota\hat{\mathbf{w}}\theta}$, a much simpler representation than $4 \times 4$ matrices, also giving a single transformation operation for coordinates, momentum and fields, and so not requiring multiple transformation relations as is conventionally required. Energy and momentum conservation is described within our description, using the single concept of momentum multivector conservation, which we illustrate through recovering Compton's formula for a photon-electron interaction.

Clifford's geometric algebra is a general mathematical formalism, which can be applied to many areas of physics, and because it can represent complex-like numbers using the trivector, and quaternion-like numbers through the even subalgebra, it can describe many phenomena, including quantum mechanics over a strictly real three-dimensional space with obvious advantages in interpretation.

## Appendix A. Geometric product

Given two vectors $\mathbf{u} = u_1e_1 + u_2e_2 + u_3e_3$ and $\mathbf{v} = v_1e_1 + v_2e_2 + v_3e_3$, using the distributive law for multiplication over addition, as assumed for an algebraic field, we find their product

$$\begin{aligned} \mathbf{uv} &= (u_1e_1 + u_2e_2 + u_3e_3)(v_1e_1 + v_2e_2 + v_3e_3) \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &+ i((u_2v_3 - u_3v_2)e_1 + (u_3v_1 - u_1v_3)e_2 + (u_1v_2 - u_2v_1)e_3), \end{aligned} \tag{A.1}$$

where we have used the elementary properties of $e_1, e_2, e_3$ defined in Eq. (1). We recognize $u_1v_1+u_2v_2+u_3v_3$ as the conventional dot product and $(u_2v_3-u_3v_2)e_1+(u_3v_1-u_1v_3)e_2+(u_1v_2 - u_2v_1)e_3$ as the cross product, so that we can write

$$\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + i\mathbf{u}\times\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}, \tag{A.2}$$

with the identity $i\mathbf{u}\times\mathbf{v} = \mathbf{u}\wedge\mathbf{v}$. For $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ unit vectors, we have $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}} = \cos\theta$ and $\hat{\mathbf{u}}\wedge\hat{\mathbf{v}} = i\hat{\mathbf{w}}\sin\theta$, where $\hat{\mathbf{w}}$ is the orthogonal vector to $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, giving $\hat{\mathbf{u}}\hat{\mathbf{v}} = \cos\theta + i\hat{\mathbf{w}}\sin\theta = \mathrm{e}^{i\hat{\mathbf{w}}}$, where $\theta$ is the angle between the two vectors. We have written the geometric product in exponential form above, using the standard definition of the exponential function, and the fact that $(i\hat{\mathbf{w}})^2 = i^2\hat{\mathbf{w}}^2 = -1$.

Using the commutivity of the dot product and the anticommutivity of the cross product we can rearrange Eq. (A.2) to write

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}) \text{ and } \mathbf{u}\wedge\mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu}). \tag{A.3}$$

We can see from Eq. (A.1), that the square of a vector $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + v_3^2$, becomes a scalar quantity. Hence the Pythagorean length of a vector is simply $|\mathbf{v}| = \sqrt{\mathbf{v}^2}$, and so we can find the inverse vector

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v}^2}, \tag{A.4}$$

thus allowing division by Cartesian vectors.

## Appendix B. Rotations in space

Rotations can be described as a sequence of two reflections. Given a vector $\mathbf{n}_1$ normal to a reflecting surface, with an incident ray given by $\mathbf{I}$, then we find the reflected ray [26]

$$\mathbf{r} = -\mathbf{n}_1\mathbf{I}\mathbf{n}_1. \tag{B.1}$$

If we apply a second reflection, with a unit normal $\mathbf{n}_2$, then we have

$$\mathbf{r} = \mathbf{n}_2\mathbf{n}_1\mathbf{I}\mathbf{n}_1\mathbf{n}_2 = \mathrm{e}^{-i\hat{\mathbf{u}}\theta}\mathbf{I}\mathrm{e}^{i\hat{\mathbf{u}}\theta}, \tag{B.2}$$

using Eq. (A.2) for two unit vectors. If the two normals $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel, then no rotation is produced. In fact the rotation produced is twice the angle between the two normals.

Hence rotations are naturally produced by conjugation, where if we seek to rotate a vector $\mathbf{v}$ by an angle $\theta$, we calculate

$$\mathbf{v}' = \mathrm{e}^{-i\hat{\mathbf{u}}\theta/2}\mathbf{v}\mathrm{e}^{i\hat{\mathbf{u}}\theta/2}, \tag{B.3}$$

which rotates about the axis $\hat{\mathbf{u}}$ in an anticlockwise direction, which is in the form of a quaternionic rotation as first discovered by Hamilton.

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